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On Howard's upper bound for heat transport by turbulent convection

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The variational problem introduced by Howard (1963) for the derivation of an upper bound on heat transport by convection in a layer heated from below is analyzed for the case in which the equation of continuity is added as constraint for the velocity field. Howard's conjecture that the maximizing solution of the Euler equations is characterized by a single horizontal wave-number is shown to be true only for a limited range of the Rayleigh number, Ra. A new class of solutions with a multiple boundary-layer structure is derived. The upper bound for the Nusselt number, Nu, given by these solutions is $Nu \leqslant (Ra/1035)^{\frac{1}{2}}$ in the limit when the Rayleigh number tends to infinity. The comparison of the maximizing solution with experimental observations by Malkus (1954a) and Deardorff & Willis (1967) emphasizes the similarity pointed out by Howard.

1. Introduction

The object of a theory of turbulence is to derive from the basic equations expressions which describe the mean properties of the turbulent flow in terms of externally given parameters. The basic difficulty encountered by the theoretical analysis arises from the fact that a description in terms of mean quantities necessarily has to be incomplete. The equations for the mean quantities which are obtained by corresponding averages of the basic equations have the property that there are always more dependent variables than equations. This gap of information is usually closed by the introduction of assumptions which cannot be free from arbitrariness. An alternative and perhaps more natural way is to derive bounds for the mean quantities. In this case the lack of information corresponds to the partial indeterminacy of the final result.

The latter theoretical approach to turbulence was first used by Howard (1963) in his work on turbulent convection in a layer heated from below. Following earlier ideas of Malkus (1954b) Howard analyzed a variational problem which enabled him to derive an exact upper bound for the convective heat transport. Howard's theory is of considerable importance since it can be shown (Busse 1969) that analogous bounds can be derived in a great variety of turbulent transport processes. For the derivation of the exact upper bound, Howard has used only the information contained in the two power or dissipation rate integrals of the basic Boussinesq equations. By taking into account supplementary information about the turbulent convection, the bound on the heat transport can be improved. In principle this method allows a systematic approximation of the solution which maximizes heat transport among all possible solutions of the basic equations. A most significant step in this direction is the solution of the variational problem under the additional constraint of the equation of continuity. Howard has solved the corresponding Euler equations by restricting the analysis to separable solutions. He conjectured that the unknown class of solutions for which the equations do not separate do not lead to a higher bound on the heat transport.

The present paper gives an extension of Howard's work to which we shall refer in the following as (H). By deriving a new class of non-separable solutions, it will be shown that Howard's conjecture becomes incorrect for sufficiently large Rayleigh numbers. The solutions are characterized by a structure of subsequent boundary layers in each of which the horizontal dependence is determined by a certain wave-number α_n. Howard's 'single α solution' can be regarded as the first member in this class of 'multi α solutions'. There are strong reasons, though no formal proof, that this class of solutions describes the correct upper bound on the heat transport among all possible solutions of the Euler equations.

The paper starts with the formulation of the variational problem in § 2. The difficulty of solving the variational problem originates from the fact that the Euler equations allow an infinite manifold of solutions, each of which corresponds to a relative extremum of the variational functional. The solution corresponding to the absolute extremum has to be determined by additional differentiation with respect to certain parameters. The complete manifold of solutions, however, is not known. Using boundary-layer analysis we shall derive in § 3 the class of multi α solutions. The upper bound for the turbulent heat transport given by these solutions is discussed in § 4. The process by which one solution follows the other in providing the upper bound can be compared to the process of subsequent instabilities, as will be shown in § 5.

Although the extremalizing solutions of the variational problem are only mathematical tools to determine the upper bound for the heat transport, they have physical significance beyond this purpose. They are of obvious interest as solutions of the strongly non-linear Euler equations which rather closely resemble the basic Boussinesq equations. There seems to exist, however, an even more direct relation. It is well known that the static state of pure heat conduction can be considered as the solution which minimizes heat flux at a given temperature difference. The convective state seems to tend toward the other extreme. The hypothesis that the physically realized convection maximizes the heat transport among all possible solutions of the basic equations was first used in the theoretical analysis by Malkus (1954*b*). If this hypothesis is approximately correct, certain similarities can be expected between the structure of turbulent convection and the extremalizing solution of the variational problem. The experimental observations show indeed such a correlation to a remarkably high degree. Howard has demonstrated this fact by comparing the single α solution with data obtained by Townsend (1959). Recently new experiments have been performed by Deardorff & Willis (1967). It will be shown in § 6 that their measurements reflect characteristic properties of the maximizing multi α solutions.

2. The formulation of the variational problem

The idea of Howard's variational method for obtaining upper bounds on the convective heat transport is to determine the maximum of the heat transport among a manifold of functions, which contains all possible solutions of the basic equations. In place of the velocity and temperature fields satisfying the equation of motion and the heat equation, the class of 'velocity' and 'temperature' fields is considered which satisfy only certain moments of the equations. The determination of the maximum of the convective heat transport among this enlarged class of fields by a variational problem provides the upper bound on the heat transport of the physically realized convection.

The introduction to the problem of convection in a layer heated from below can be kept short, since a detailed description of the problem is given in (H) as well as in many other papers on this subject. As basic equations the Navier–Stokes equations for the velocity vector **v** and the heat equation are assumed in the form of the Boussinesq approximation. We shall use a Cartesian system of co-ordinates with the origin in the centre between the rigid infinitely extended parallel plates which bound the fluid. The direction of the z co-ordinate is chosen perpendicular to the plates opposite to the direction of gravity. The temperature field is divided into two parts, a part T depending on z only and equal to the horizontal average of the temperature, and the remaining fluctuating part θ. The temperatures at the boundaries are given constants, T_1 and T_2, corresponding to the assumption of infinitely conducting plates. In the usual dimensionless form with the distance d between the plates, d^2/κ, $(T_2-T_1)/Ra$, as the basic scales for length, time and temperature respectively, the Boussinesq equations are:

$$\left.\begin{aligned}\Delta\mathbf{v}+\boldsymbol{\lambda}\theta-\nabla\omega &= \frac{\kappa}{\nu}\left(\mathbf{v}.\nabla\mathbf{v}+\frac{\partial}{\partial t}\mathbf{v}\right),\\ \nabla.\mathbf{v} &= 0,\\ \Delta\theta-w\frac{\partial T}{\partial z} &= \mathbf{v}.\nabla\theta-\overline{\mathbf{v}.\nabla\theta}+\frac{\partial}{\partial t}\theta,\\ \frac{\partial^2 T}{\partial z^2} &= \frac{\partial}{\partial z}\overline{w\theta}+\frac{\partial}{\partial t}T.\end{aligned}\right\} \tag{2.1}$$

The bar indicates the average with respect to the horizontal co-ordinates. $\boldsymbol{\lambda}$ is the unit vector in the direction of the z co-ordinate; w is the z component of the velocity vector.

In accordance with the separation of the temperature field into its two parts θ and T, and the absence of a mean velocity component, the following property has to be satisfied:

The functions **v**, θ are bounded, and the horizontal averages of products of these functions like $w\theta$ do exist. (2.2)

We shall use an additional property which can be considered as part of the definition of turbulent convection in a stationary state:

The horizontal averages, in particular $\overline{w\theta}$, are time independent. (2.3)

This property allows the integration of the last equation of (2.1),

$$\frac{dT}{dz} = -Ra - \langle w\theta \rangle + \overline{w\theta}, \tag{2.4}$$

with the angle brackets denoting the average over the entire fluid layer. The boundary conditions determine the constant of integration, $Ra + \langle w\theta \rangle$, which is the dimensionless heat flux traversing the layer.

By multiplying the first equation of (2.1) by $\mathbf{v}$ and the third equation by θ and averaging the equations over the fluid layer, the first moments of the equations are obtained which are called 'power integrals' in Howard's paper:

$$\left.\begin{aligned} \langle |\nabla \times \mathbf{v}|^2 \rangle &= \langle w\theta \rangle, \\ \langle |\nabla\theta|^2 \rangle &= Ra\langle w\theta \rangle - \langle (\overline{w\theta} - \langle w\theta \rangle)^2 \rangle. \end{aligned}\right\} \tag{2.5}$$

In deriving these relations the equation of continuity has been used to transform the advection terms into surface integrals. These integrals vanish owing to the condition $w = 0$ at the boundary and to the fact that the contributions from the surface which bounds the fluid at infinity with respect to the horizontal dimensions can be neglected in comparison with the volume integrals. Relations (2.5) differ from the corresponding relations (7) and (8) in (H) only in respect that θRa replaces Howard's T.

The variational problem is to determine the maximum of the convective heat transport $\langle w\theta \rangle$ at a given Rayleigh number Ra. As competitors all fields $\mathbf{v}$, θ are admitted that vanish at $z = \pm\frac{1}{2}$, possess the properties (2.2) and (2.3), and satisfy the equation of continuity and relations (2.5). They do not have to satisfy, however, the Boussinesq equations. Equations (2.1) aside from the continuity equation do not enter into the problem being considered in this paper except in the derivation of (2.5). For convenience we use the same symbols $\mathbf{v}$, θ as before, keeping in mind that they represent generalized fields which no longer have the physical meaning of velocity vector and temperature.

Instead of asking for the maximum μ of $\langle w\theta \rangle$ at a given value of the Rayleigh number, we can ask for the minimum R of the Rayleigh number as a functional of $\mathbf{v}$, θ at a given value μ of $\langle w\theta \rangle$. The equivalence of both problems can be shown by proving that $R(\mu)$ is a bounded monotone function for all finite values of μ. The second problem allows a particularly simple formulation:

Given $\mu > 0$, find the minimum $R(\mu)$ of the functional

$$\mathscr{R}(\mathbf{v}, \theta, \mu) \equiv \frac{\langle |\nabla \times \mathbf{v}|^2 \rangle \langle |\nabla\theta|^2 \rangle}{\langle w\theta \rangle^2} + \mu \frac{\langle (\overline{w\theta} - \langle w\theta \rangle)^2 \rangle}{\langle w\theta \rangle^2} \tag{2.6}$$

among all fields $\mathbf{v}$, θ *that vanish at* $z = \pm\frac{1}{2}$, *satisfy the equation of continuity, and have the properties* (2.2), (2.3).

Since the functional (2.6) is homogeneous of degree zero with respect to $\mathbf{v}$ as well as to θ, the amplitude of both quantities is left undetermined for any solution of the variational problem. This fact allows us to impose

$$\langle w\theta \rangle = \mu, \quad \langle |\nabla \times \mathbf{v}|^2 \rangle = \mu, \tag{2.7}$$

as normalization conditions. Thus the first of relations (2.5) is automatically satisfied, while the second is easily recognized in the definition (2.6), where

$\mathscr{R}(\mathbf{v}, \theta, \mu)$ stands for the Rayleigh number Ra considered as a functional of $\mathbf{v}$, θ, and μ. The formulation of the variational problem given above differs slightly from the formulation $P2$ in (H). $\mathscr{R}/\mu$ and μ^{-1} correspond to the variables $\mathscr{F}$ and λ respectively used in (H). The proof for the monotonicity of $R(\mu)$ follows in complete analogy to the proof given by Howard.

Another step in the simplification of the variational problem is the elimination of the continuity equation as a side condition. An arbitrary vector field $\mathbf{v}$ satisfying $\nabla . \mathbf{v} = 0$ can be represented by

$$\mathbf{v} = \nabla \times (\nabla \times \boldsymbol{\lambda}) v + \nabla \times \boldsymbol{\lambda} \psi. \tag{2.8}$$

The vertical component of the vector field $\mathbf{v}$ in this representation is

$$w = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) v \equiv -\Delta_2 v \tag{2.9}$$

and the dissipation term can be written

$$\langle |\nabla \times \mathbf{v}|^2 \rangle = -\langle v \Delta \Delta \Delta_2 v \rangle + \langle \psi \Delta \Delta_2 \psi \rangle. \tag{2.10}$$

Since w does not depend on ψ the minimum of the functional (2.6) obviously is reached in the case $\psi = 0$, i.e. when the vertical component of vorticity of $\mathbf{v}$ vanishes. Thus the number of varying functions in (2.6) has been reduced to v and θ.

Since the normalization condition (2.7) does not enter the actual variational problem, we replace it for the following discussion by the more convenient condition

$$\left.\begin{aligned} \langle w\theta \rangle &= 1, \\ \langle ww \rangle &= \langle \theta\theta \rangle. \end{aligned}\right\} \tag{2.11}$$

3. Boundary-layer solutions of the variational problem

The homogeneity of the variational problem (2.6) with respect to the horizontal dimensions suggests that the solutions of the corresponding Euler equations have an x, y dependence in the form of waves. We assume for this reason

$$v = v^{(N)} \equiv \sum_{n=1}^{N} \phi_n(x,y) \frac{w_n(z)}{\alpha_n^2}, \quad \theta = \theta^{(N)} \equiv \sum_{n=1}^{N} \phi_n(x,y)\, \theta_n(z), \tag{3.1}$$

where $\phi_n(x,y)$ satisfies the relations

$$\left.\begin{aligned} \Delta_2 \phi_n(x,y) &= -\alpha_n^2 \phi_n(x,y) \\ \text{and} \qquad \langle \phi_n^2 \rangle &= 1. \end{aligned}\right\} \tag{3.2}$$

The functions ϕ_n, w_n, θ_n and α_n will in general depend on the choice of N. In this section, however, N is regarded as a given arbitrary integer and the indication of the dependence of N will be neglected. In the case $N = 1$, (3.1) reduces to the separable form of solutions considered by Howard.

Following the analysis in (H) we shall use boundary-layer methods to solve the variational problem in the limit when μ tends to infinity. Since in this limit the second term on the right-hand side of (2.6) becomes dominant, it can be assumed that the integrand in

$$\langle(\overline{w\theta}-\langle w\theta\rangle)^2\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} (\overline{w\theta}-\langle w\theta\rangle)^2\,dz \tag{3.3}$$

vanishes throughout the interior of the layer and contributes to the integral only in regions with a thickness of the order μ^{-r} adjacent to the planes $z = \pm\frac{1}{2}$. r is a positive number to be determined. Close to the boundary a rapid change has to occur in the z dependence of $\mathbf{v}$ and θ. Since the dissipation term in functional (2.6) in general assumes a minimum when the horizontal scale of $\mathbf{v}$, θ is comparable to the vertical scale, we allow for a sequence of different boundary layers by which the horizontal scale is adjusted from its interior value to its value at the boundary. The thickness of each boundary layer is supposed to be large in comparison to the thickness of the following layer, and $\overline{w\theta}$ is supposed to be approximately equal to one in all but the last of the boundary layers, where it is still $o(1)$. Accordingly we introduce boundary-layer co-ordinates

$$\zeta_n = \mu^{r_n}(\tfrac{1}{2}\mp z) \tag{3.4}$$

for $n = 1, \ldots, N$ with $r_N = r$. We assume that w_n, θ_n differ from zero essentially only in two successive boundary layers,

$$w_n(z) = \mu^{-s_n}\tilde{w}_n(\zeta_{n-1}), \quad \theta_n(z) = \mu^{s_n}\tilde{\theta}_n(\zeta_{n-1}) \quad \text{for} \quad (\tfrac{1}{2}\mp z) \approx o(\mu^{-r_{n-1}}), \tag{3.5}$$

$$w_n(z) = \mu^{-p_n}\hat{w}_n(\zeta_n), \quad \theta_n(z) = \mu^{p_n}\hat{\theta}_n(\zeta_n) \quad \text{for} \quad (\tfrac{1}{2}\mp z) \approx o(\mu^{-r_n}). \tag{3.6}$$

For all but the last boundary layer we have

$$\hat{w}_n\hat{\theta}_n + \tilde{w}_{n+1}\tilde{\theta}_{n+1} \approx 1. \tag{3.7}$$

We set

$$\alpha_n^2 = \mu^{q_n} b_n^2 \tag{3.8}$$

and assume that b_n^2 as well as $\hat{w}_n$, $\tilde{w}_n$, $\hat{\theta}_n$, $\tilde{\theta}_n$ are quantities of the order one independent of μ. The boundary-layer approximation of the right-hand side of (2.6) yields

$$\begin{aligned}
\hat{\mathscr{R}}(v^{(N)},\theta^{(N)};\mu) = {} & \mu^{1-r_N}\int_0^\infty (1-\hat{w}_N\hat{\theta}_N)^2\,d\zeta_N \\
& + \left\{2\sum_{n=1}^{N}\frac{\mu^{3r_n-2p_n-q_n}}{b_n^2}\int_0^\infty \hat{w}_n''^2\,d\zeta_n + 2\sum_{n=2}^{N} b_n^2\mu^{q_n-r_{n-1}-2s_n}\right. \\
& \qquad \left.\times\int_0^\infty \tilde{w}_n^2\,d\zeta_{n-1} + \mu^{q_1}b_1^2\right\} \\
& \times\left\{2\sum_{n=1}^{N}\mu^{r_n+2p_n}\int_0^\infty \hat{\theta}'^2\,d\zeta_n + 2\sum_{n=2}^{N} b_n^2\mu^{q_n-r_{n-1}+2s_n}\right. \\
& \qquad \left.\times\int_0^\infty \tilde{\theta}_n^2\,d\zeta_{n-1} + \mu^{q_1}b_1^2\right\}.
\end{aligned} \tag{3.9}$$

In order to distinguish the boundary-layer approximation from the exact functional $\mathscr{R}(v^{(N)},\theta^{(N)};\mu)$ we have denoted it by the superscript $\wedge$. In addition to the terms neglected according to the boundary-layer assumption, we have neglected

$$4\sum_{n=1}^{N}\mu^{r_n-2p_n}\int_0^\infty \hat{w}_n'^2\,d\zeta_n, \tag{3.10}$$

which represents the last term in the following decomposition of the viscous dissipative term

$$-\langle v\Delta\Delta\Delta_2 v\rangle = \left\langle \left|\boldsymbol{\lambda}\times\nabla\frac{\partial^2}{\partial z^2}v\right|^2\right\rangle + \langle|\boldsymbol{\lambda}\times\nabla\Delta_2 v|^2\rangle + 2\left\langle\left|\frac{\partial}{\partial z}\Delta_2 v\right|^2\right\rangle. \quad (3.11)$$

According to Schwarz's inequality the last term on the right side of (3.11) cannot exceed the sum of the first and second terms. For this reason neglecting (3.10) cannot change the order of magnitude of the minimum $\hat{R}^{(N)}(\mu)$ of $\hat{\mathscr{R}}(v^{(N)}, \theta^{(N)}; \mu)$ which will be determined in the next paragraph. It will turn out subsequently that (3.10) is in fact of smaller order of magnitude than the other terms in the first wavy bracket of (3.9)

The functional (3.9) reaches its minimum as a function of the variables r_n, q_n, p_n and s_n when the maximum of the exponents of μ in the expression on the right-hand side of (3.9) reaches its minimum. This minimum corresponds to the case when as many as possible of the exponents attain the same maximal value Since there are $4N-1$ independent variables r_n, q_n, p_n and s_n, at least $4N$ exponents can be chosen to be equal. The particular choice of the following $4N-1$ equations

$$\left.\begin{aligned} 1-r_N = 4r_N - q_N = 4r_{N-1}-q_{N-1} = \ldots = 4r_1 - q_1 \\ = 2q_N - 2r_{N-1} = 2q_{N-1}-2r_{N-2} = \ldots = 2q_1; \\ \left.\begin{aligned} 3r_n - 2(p_n - s_n) - r_{n-1} = r_n + q_n + (2p_n - s_n) - r_{n-1}, \\ q_n - r_{n-1} - 2s_n - q_1 = q_n - r_{n-1} + 2s_n - q_1, \end{aligned}\right\} \quad (n=2,\ldots,N); \\ 3r_1 - 2p_1 = r_1 + q_1 + 2p_1 \end{aligned}\right\} \quad (3.12)$$

yields as solutions

$$r_n = \frac{1-4^{-n}}{3-4^{-N}}, \quad q_n = \frac{2-4^{-n+1}}{3-4^{-N}}, \quad s_n = 0, \quad 2p_n = \frac{4^{-n}}{3-4^{-N}}. \quad (3.13)$$

Thus $\hat{R}(\mu)$ is of the order $\mu^{2/(3-4^{-N})}$ and all terms in the wavy brackets are of the order $\mu^{1/(3-4^{-N})}$. The choice of equations (3.12) has been induced by the fact that all variables occur independently with both signs. Hence any deviation from the solution (3.13) will lead to an exponent larger than $2/(3-4^{-N})$.

The functions $\hat{w}_n$, $\tilde{w}_n$, $\hat{\theta}_n$, $\tilde{\theta}_n$ will be determined by the Euler equations which are the necessary conditions for an extremum of $\hat{\mathscr{R}}(v^{(N)}, \theta^{(N)}; \mu)$,

$$\left.\begin{aligned} D_\theta \hat{w}_n^{\text{iv}}/b_n^2 - \mu^{r_N - r_n}(1-\hat{w}_n\hat{\theta}_n - \tilde{w}_{n+1}\tilde{\theta}_{n+1})\hat{\theta}_n = 0 \\ D_w\hat{\theta}_n'' - \mu^{r_N-r_n}(1-\hat{w}_n\hat{\theta}_n - \tilde{w}_{n+1}\tilde{\theta}_{n+1})\hat{w}_n = 0 \end{aligned}\right\} \quad (n=1,\ \ldots,\ N), \quad (3.14)$$

$$\left.\begin{aligned} D_\theta b_{n+1}^2 \tilde{w}_{n+1} - \mu^{r_N-r_n}(1-\hat{w}_n\hat{\theta}_n - \tilde{w}_{n+1}\tilde{\theta}_{n+1})\tilde{\theta}_{n+1} = 0 \\ D_w b_{n+1}^2\tilde{\theta}_{n+1} - \mu^{r_N-r_n}(1-\hat{w}_n\hat{\theta}_n - \tilde{w}_{n+1}\tilde{\theta}_{n+1})\tilde{w}_{n+1} = 0 \end{aligned}\right\} \quad (n=1,\ \ldots,N). \quad (3.15)$$

Here and in the following discussion $\tilde{w}_{N+1}$, $\tilde{\theta}_{N+1}$ are to be replaced by zero. As abbreviations

$$D_w \equiv \left[\sum_{n=1}^N \frac{1}{b_n^2}\int_0^\infty (\hat{w}_n'')^2\, d\zeta_n + \sum_{n=2}^N b_n^2 \int_0^\infty \tilde{w}_n^2\, d\zeta_{n-1} + b_1^2\right], \quad (3.16)$$

$$D_\theta \equiv \left[\sum_{n=1}^N \int_0^\infty (\hat{\theta}_n')^2\, d\zeta_n + \sum_{n=2}^N b_n^2 \int_0^\infty \tilde{\theta}_n^2\, d\zeta_{n-1} + b_1^2\right] \quad (3.17)$$

have been introduced. A consequence of (3.14) and (3.15) is

$$\left.\begin{aligned} D_\theta\langle\hat{w}''^2\rangle/b_n^2 &= D_w\langle\hat{\theta}_n'^2\rangle,\\ D_\theta\tilde{w}_{n+1}^2 &= D_w\tilde{\theta}_{n+1}^2,\end{aligned}\right\} \tag{3.18}$$

from which

$$D_\theta(D_w-b_1^2) = D_w(D_\theta-b_1^2),$$

and hence

$$D_\theta = D_w \equiv D$$

can be concluded since $\hat{\mathscr{R}}$ diverges if b_1^2 vanishes. According to (3.18) $\tilde{w}_{n+1}$ is equal to $\tilde{\theta}_{n+1}$ and as long as it is non-vanishing

$$(1-\hat{w}_n\hat{\theta}_n-\tilde{w}_{n+1}\tilde{\theta}_{n+1}) = \mu^{r_n-r_N}b_{n+1}^2D \tag{3.19}$$

holds for $n = 1, \ldots, N-1$. This relation indicates that relation (3.7) cannot be satisfied exactly. The contribution of (3.19) in the integral (3.3), however, is negligible. Equation (3.19) can be used to replace the corresponding term in (3.14) as long as $\hat{w}_n\hat{\theta}_n$ differs from one. In the region where $\hat{w}_n\hat{\theta}_n$ has approached unity, $\tilde{w}_{n+1}$ and $\tilde{\theta}_{n+1}$ vanish and (3.19) no longer holds. In this case the condition $\hat{w}_n\hat{\theta}_n = 1$ allows us to replace (3.14) by

$$\frac{1}{b_n^2}\hat{w}_n^{\mathrm{iv}} = \frac{\hat{\theta}_n\hat{\theta}_n''}{\hat{w}_n} = \frac{\hat{w}_n''\hat{w}_n-\hat{w}_n'^2}{\hat{w}_n^5}. \tag{3.20}$$

In the region described by this equation w_n has to tend to infinity in order to join $\mu^{p_n}\tilde{w}_n$ while $\hat{w}_n''$ and $\hat{w}_n'''$ have to tend to zero in order to yield finite values for the integrals occurring on the right side of (3.16). This condition together with the condition

$$\hat{w}_n = \hat{w}_n' = \hat{\theta}_n = 0 \quad \text{at} \quad \zeta_n = 0 \tag{3.21}$$

will suffice to determine the solution of (3.14), (3.20). By introducing new variables

$$\left.\begin{aligned} \xi &= \zeta_n b_n^{\frac{1}{3}} b_{n+1}^{\frac{2}{3}},\\ \hat{\Omega} &= \hat{w}_n b_n^{-\frac{1}{3}} b_{n+1}^{\frac{1}{3}},\\ \hat{\Theta} &= \hat{\theta}_n b_n^{\frac{1}{3}} b_{n+1}^{-\frac{1}{3}},\end{aligned}\right\} \tag{3.22}$$

(3.14) and (3.20) can be transformed into

$$\left.\begin{aligned} \hat{\Omega}^{\mathrm{iv}}-\hat{\Theta} &= 0,\\ \hat{\Theta}''+\hat{\Omega} &= 0,\end{aligned}\right\} \tag{3.23}$$

and

$$\hat{\Omega}^{\mathrm{iv}} = \frac{\hat{\Omega}''\hat{\Omega}-2\hat{\Omega}'^2}{\hat{\Omega}^5}, \tag{3.24}$$

respectively. The solution of (3.23) and (3.24) together with the corresponding boundary conditions is given in the appendix. In the following discussion we only need the value

$$3\beta \equiv \int_0^\infty \hat{\Omega}''^2 d\xi + \int_0^\infty (1-\hat{\Omega}\hat{\Theta})\, d\xi = 1{\cdot}847. \tag{3.25}$$

In the case $n = N$ the factor of $\hat{\theta}_n$, $\hat{w}_n$ in (3.14) cannot be replaced by a constant and a change of variables similar to (3.17) with D in place of b_{n+1}^{-2} leads to the equations

$$\left.\begin{aligned} \Omega^{\mathrm{iv}}-(1-\Omega\Theta)\,\Theta &= 0,\\ \Theta''+(1-\Omega\Theta)\,\Omega &= 0.\end{aligned}\right\} \tag{3.26}$$

This system of equations together with the corresponding boundary conditions has been solved in (H) with the result

$$\sigma \equiv \int_0^\infty \Omega''^2 \, d\xi = \int_0^\infty \Theta'^2 \, d\xi = \frac{1}{4}\int_0^\infty (1-\Omega\Theta)^2 \, d\xi = 0{\cdot}337. \tag{3.27}$$

The last step in the solution of the problem is the determination of the variables b_n^2. Using (3.13), (3.25), (3.27) in (3.9) we get

$$\hat{\mathscr{R}}(\hat{v}^{(N)}, \hat{\theta}^{(N)}; \mu) = \mu^{2/(3-4^{-N})}\{b_N^{-\frac{1}{3}} D^{\frac{1}{3}} 8\sigma + D^2\}, \tag{3.28}$$

with
$$D = \sum_{n=1}^{N-1} 6\beta \left[\frac{b_{n+1}^4}{b_n}\right]^{\frac{1}{3}} + \frac{2\sigma}{(b_N D^2)^{\frac{1}{3}}} + b_1^2.$$

By differentiation of $\hat{\mathscr{R}}$ with respect to b_n we obtain as the necessary condition for a minimum

$$\frac{\partial D}{\partial b_1} = 0 \curvearrowright \beta\left[\frac{b_2}{b_1}\right]^{\frac{4}{3}} - b_1 = 0,$$

$$\frac{\partial D}{\partial b_n} = 0 \curvearrowright \left[\frac{b_{n+1}}{b_n}\right]^{\frac{4}{3}} - 4\left[\frac{b_n}{b_{n-1}}\right]^{\frac{1}{3}} = 0 \quad \text{for} \quad 1 < n < N,$$

$$\frac{\partial \hat{\mathscr{R}}}{\partial b_N} = 0 \curvearrowright \frac{\sigma}{(D^2 b_N^4)^{\frac{1}{3}}} - 4\beta\left[\frac{b_N}{b_{N-1}}\right]^{\frac{1}{3}} = 0.$$

After simple calculations these relations yield

$$\left.\begin{aligned} b_1^{4(3-4^{-N})} &= 4^{-6N}(\sigma/\beta)^3(\beta 4^{\frac{4}{3}})^{4(1-4^{-N})}(1-4^{-N})^{-2}, \\ b_n &= b_1 4^{n-1}\left[\frac{b_1 4^{\frac{4}{3}}}{\beta}\right]^{1-4^{-n+1}} \quad \text{for} \quad n \geqslant 1. \end{aligned}\right\} \tag{3.29}$$

Using these results the minimum $\hat{R}^{(N)}(\mu)$ of $\hat{\mathscr{R}}(v^{(N)}, \theta^{(N)}; \mu)$ can be expressed by

$$\hat{R}^{(N)}(\mu) = (3-4^{-N})/(1-4^{-N})\, D^2 \mu^{2/(3-4^{-N})}, \tag{3.30}$$

with
$$D = (4^N - 1)\, b_1^2,$$

which represents the main result of this section.

We note that the solution $v^{(N)}, \theta^{(N)}$ in fact satisfies the normalization condition $-\langle \theta^{(N)} \Delta_2 v^{(N)} \rangle = 1$ assumed in the definition (3.9) of $\hat{\mathscr{R}}$. The proof follows readily by the following consideration. By multiplying the first equations of (3.14) and (3.15) by $\hat{w}_n$ and $\tilde{w}_{n+1}$ respectively and integrating the equations over ζ_n and using (3.7), we obtain

$$\begin{aligned} \mu^{r_N - r_n} D^{-1} \int_0^\infty (1 - \tilde{w}_{n+1}\tilde{\theta}_{n+1} - \hat{w}_n \hat{\theta}_n)\, d\zeta_n &= b_n^{-2}\int_0^\infty \hat{w}_n''^2 \, d\zeta_n + b_{n+1}^2 \int_0^\infty \tilde{w}_{n+1}^2 \, d\zeta_n \\ &= 3\beta\left[\frac{b_{n+1}^4}{b_n}\right]^{\frac{1}{3}}, \end{aligned} \tag{3.31}$$

for all n less than N. The corresponding integral for $n = N$ with vanishing $\tilde{w}_{N+1}$, $\tilde{\theta}_{N+1}$ can be evaluated using the property

$$\int_0^\infty (1 - \Omega\Theta)\, d\xi = 5\sigma \tag{3.32}$$

of the equations (3.26), (3.27). Finally, the deviation from unity of $\tilde{w}_1(z)$, $\tilde{\theta}_1(z)$ has to be determined by the corresponding Euler equations

$$D b_1^2 \mu^{2/(3-4^{-N})} \tilde{w}_1 - \mu(1-\tilde{w}_1\tilde{\theta}_1)\tilde{\theta}_1 - \hat{R}\tilde{\theta}_1 = 0,$$
$$D b_1^2 \mu^{2/(3-4^{-N})} \tilde{\theta}_1 - \mu(1-\tilde{w}_1\tilde{\theta}_1)\tilde{w}_1 - \hat{R}\tilde{w}_1 = 0.$$

The solution of these equations gives $\tilde{w}_1 = \tilde{\theta}_1$ as has been concluded before and

$$\tilde{w}_1\tilde{\theta}_1 - 1 = \mu^{-1}(\hat{R} - b_1^2 D \mu^{2/(3-4^{-N})}) = \mu^{-r_N} 3(4^N - \tfrac{2}{3}) b_1^2 D. \tag{3.33}$$

Using (3.31), (3.32) and (3.33) it can now be shown that

$$\left\langle 1 - \sum_{n=1}^{N} w_n \theta_n \right\rangle$$

is vanishing:

$$\left\langle 1 - \sum_{n=1}^{N} w_n \theta_n \right\rangle = \sum_{n=1}^{N-1} 2\mu^{-r_n} \int_0^\infty (1 - \tilde{w}_{n+1}\tilde{\theta}_{n+1} - \hat{w}_n\hat{\theta}_n)\, d\zeta_n$$
$$+ 2\int_0^\infty (1 - \hat{w}_N\hat{\theta}_N)\, d\zeta_N - \tilde{w}_1\tilde{\theta}_1 + 1$$
$$= \left\{ 2\sum_{n=1}^{N-1} \left[\frac{b_{n+1}^4}{b_n}\right]^{\frac{1}{3}} 3\beta + 2(b_N D^2)^{-\frac{1}{3}} 5\sigma - (3\cdot 4^N - 2) b_1^2 \right\} D\mu^{-r_N}$$
$$= 0.$$

4. The upper bound for the convective heat transport

Before we apply the solutions obtained in the preceding section to produce the upper bound for the convective heat transport, we wish to draw attention to a special feature of the variational problem. The upper bound $\mu(Ra)$ for the convective heat transport and the fact that $\langle w\theta \rangle$ has to be positive provide a necessary condition for the realized convection. It is a remarkable property that the necessary condition coincides with the sufficient condition in the limit when μ tends to zero. In this case the second term in the definition (2.6) of the functional $\mathscr{R}$ vanishes and the variational problem reduces to the variational problem by which the critical Rayleigh number R_c for the onset of infinitesimal instabilities in the static fluid layer is determined. It is well known that the minimum $R(0) = R_c$ of the functional $\mathscr{R}$ in this limit is attained by a solution of the form (3.1) with $N = 1$. Howard has used the known solution to determine also the derivative of $R(\mu)$ with respect to μ in the limit $\mu = 0$:

$$R(0) = 1707{\cdot}8, \quad \left.\frac{dR(\mu)}{d\mu}\right|_{\mu=0} = 0{\cdot}6919. \tag{4.1}$$

Since $R(\mu)$ is a monotone function, a fairly good description of the function $R(\mu)$ is given, if its behaviour for large values of μ is known. In the preceding section we have calculated minima of the functional (2.6) in the limit $\mu \to \infty$ for solutions of the form (3.1). Since the general class of solutions leading to a relative minimum of the functional (2.6) is not known, we make the assumption that the absolute minimum of the functional can be found among the class of relative minima $\{R^{(N)}(\mu)\}$ corresponding to solutions of the form (3.1) with a multiple boundary-layer structure.

Owing to the asymptotic analysis, the functions $\hat{R}^{(N)}(\mu)$ derived in § 3 coincide with the exact minima $R^{(N)}(\mu)$ only in the limit when μ tends to infinity. We expect, however, that for large but finite values of μ a reasonable approximation of $R^{(N)}(\mu)$ will be provided by $\hat{R}^{(N)}(\mu)$. It can be seen from the representation (3.30) that the absolute minimum $\hat{R}(\mu)$ among the class of functions $\{\hat{R}^{(N)}(\mu)\}$ is attained by one after the other starting with $\hat{R}^{(1)}(\mu)$. Thus $\hat{R}(\mu)$ is a smooth function of μ except for an infinite number of kinks which occur when $\hat{R}^{(N+1)}(\mu)$ drops below $\hat{R}^{(N)}(\mu)$.

For comparison with the experimental data it is convenient to draw the inverse function $\hat{\mu}(R)$ of $\hat{R}(\mu)$ divided by R, which corresponds to the absolute maximum among the class of function $\{\hat{\mu}^{(N)}(R)/R\}$. $\hat{\mu}(R)/R$ gives an upper bound for the ratio between the convective part of the heat transport H_c and the conductive part H_s. This ratio differs by one from the Nusselt number which is defined as the ratio between the total heat flux $H_s + H_c$ traversing the convective layer and the heat transport H_s in the static layer corresponding to reversed temperatures at the boundaries,

$$Nu \equiv 1 + H_c/H_s.$$

In figure 1 the functions $\hat{\mu}^{(N)}(Ra)/Ra$ have been drawn only for $N = 1, 2, 3$ because the curves become indistinguishable for larger values of N.

It is very likely, although difficult to prove, that the boundary-layer approximation $\hat{R}^{(N)}(\mu)$ is smaller than the corresponding exact dependence $R^{(N)}(\mu)$ since many positive terms have been neglected in the definition of the functional (3.9). An approximation for $R^{(N)}(\mu)$ from above is given, of course, by the value of (2.6) for an arbitrary trial field of the form (3.1). A convenient field for this purpose is the boundary-layer solution $v^{(N)}$, $\theta^{(N)}$ derived in the last section. In the earlier version of this paper (Busse 1968) all terms of smaller order which have been neglected in the expression (3.9) were evaluated. In figure 1 the resulting lower bounds for the exact dependences of $\mu^{(N)}(Ra)/Ra$ have been indicated by dashed lines. The plot indicates that the boundary-layer approximation does not give a good description of the positions of the kinks in the exact upper bound $\mu(Ra)/Ra$ whereas the quantitative approximation of this function by $\hat{\mu}(Ra)/Ra$ is reasonably close.

In the limit when μ tends to infinity the minimum $\hat{R}(\mu)$ among the class $\{\hat{R}^{(N)}(\mu)\}$ approaches

$$\hat{R}^{(\infty)}(\mu) = 3\sigma(\mu^2\beta 4^{\frac{16}{3}})^{\frac{1}{3}} = 10{\cdot}114\mu^{\frac{2}{3}}. \tag{4.2}$$

This asymptotic minimum differs only by a factor of about 3 from the minimum of functional (2.6) in the case when the equation of continuity is neglected as constraint. According to (H) the exact asymptotic value of the latter minimum is given by

$$R_1(\mu) = 4{\cdot}3^{-\frac{1}{3}}\mu^{\frac{2}{3}}. \tag{4.3}$$

The upper bound corresponding to this function is also plotted in figure 1.

It is of interest to discuss in more detail the structure of the multi α solutions $v^{(N)}$, $\theta^{(N)}$. In figure 2 the wavelengths $l_n^{(N)} = 2\pi/\alpha_n^{(N)}$ have been drawn as functions of Ra starting at the point at which $\hat{\mu}^{(N)}(Ra)$ begins to describe the upper bound for the heat transport. The wavelengths as well as the thicknesses

$$d_n^{(N)} = \mu^{-r_n}(b_n b_{n+1}^2)^{-\frac{1}{3}} \tag{4.4}$$

of the corresponding boundary layers become shorter with increasing Ra. Since the solutions tend to adjust the horizontal scale to the vertical scale, $l_n^{(N)}(Ra)$ corresponds roughly to the harmonic mean between $d_n^{(N)}(Ra)$ and $d_{n-1}^{(N)}(Ra)$.

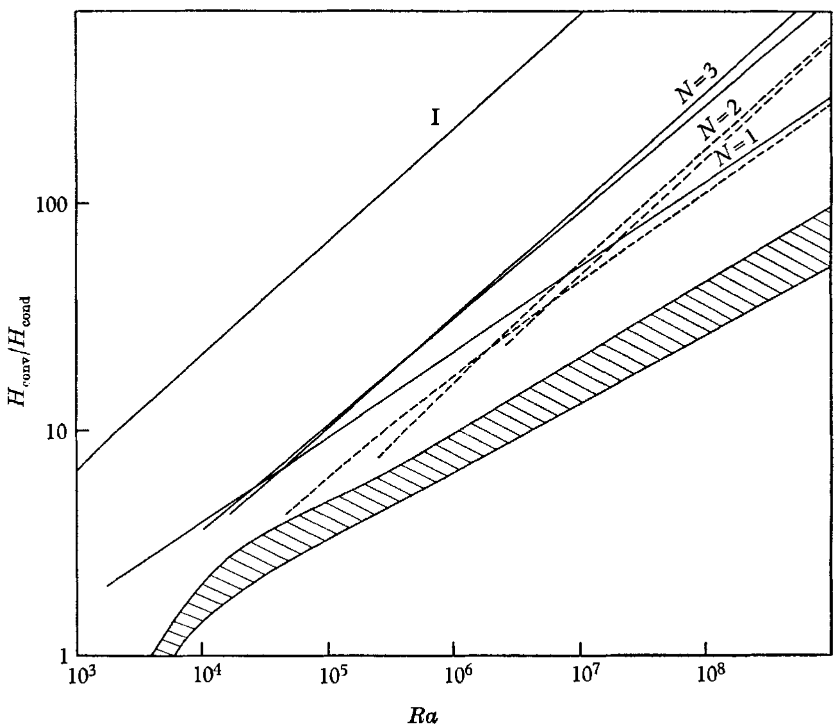

FIGURE 1. The upper bound for the Nusselt number in dependence on the Rayleigh number in comparison with experimental results (shaded region). The upper bound corresponds to the maximum among the curves $N = 1, 2, 3$. The upper bound derived without the constraint of the continuity equation is given by (I). The dashed lines give an estimate of $\mu^{(N)}(Ra)/Ra$ from below.

By the following definition we introduce the solution $v^{(e)}$, $\theta^{(e)}$ which corresponds to the minimum $\hat{R}(\mu)$,

$$\left.\begin{aligned} &(v^{(e)}, \theta^{(e)}) \equiv (v^{(N)}, \theta^{(N)}), \\ \text{whenever}\quad &\hat{R}^{(N)}(\mu) = \hat{R}(\mu) \quad \text{for} \quad N = 1, \ldots. \end{aligned}\right\} \tag{4.5}$$

This solution, which will be called the 'bounding field', is a discontinuous function of μ at the points where $\hat{R}(\mu)$ has a kink. In the limit of μ tending to infinity, however, the bounding field as well as $\hat{R}(\mu)$ can be considered as smooth functions of μ. For this purpose we eliminate the parameter N by determining the minimizing N as a function of μ. Using $x \equiv 4^{-N}$ we obtain as the condition for the minimum

$$0 = \frac{1}{\hat{R}^{(N)}(\mu)} \frac{d}{dx} \hat{R}^{(N)}(\mu) = \tfrac{1}{3} \ln\{\mu^{\frac{2}{3}} x^2 \sigma / (\beta 4^{\frac{4}{3}})^{\frac{8}{3}} \beta\}, \tag{4.6}$$

where terms which are vanishing in the limit when x tends to zero have been neglected. According to (4.6) the minimizing value of N, N_e, in the limit of large μ, is given by

$$x_e^{-1} = 4^{N_e} = (\mu\sigma^{\frac{3}{2}}\beta^{-\frac{11}{8}}4^{-\frac{16}{3}})^{\frac{1}{3}}. \tag{4.7}$$

We use this relation to evaluate the asymptotic properties of the bounding field. Its structure is characterized by the fact that the thickness of its outermost boundary layer becomes independent of μ

$$d_1^{(e)} = \mu^{-\frac{1}{4}}(b_1 b_2^2)^{-\frac{1}{3}} = \frac{1}{16\beta} = 0{\cdot}1015. \tag{4.8}$$

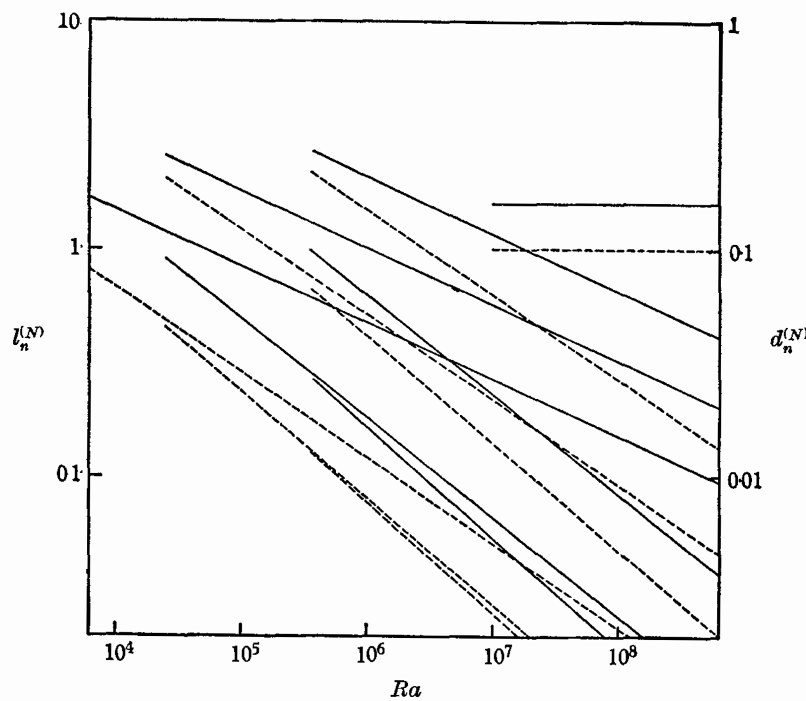

FIGURE 2. Wavelengths $l_n^{(N)} = 2\pi/\alpha_n^{(N)}$ (solid lines) and characteristic thicknesses $d_n^{(N)}$ of the boundary layers (dashed lines) for the multi α solutions.

The same fact holds for the wavelength of the interior

$$l_1^{(e)} = \frac{2\pi}{\alpha_1^{(e)}} = \frac{2\pi}{\beta 4^{\frac{1}{3}}} = 1{\cdot}606. \tag{4.9}$$

Both values (4.8) and (4.9) have been indicated in figure 2. It is apparent that the asymptotic form of the bounding field holds well even for $N_e = 2, 3$, when it is kept in mind that (4.8) and (4.9) have to be regarded as average values of the functions $l_1^{(N)}(R)$ and $d_1^{(N)}(R)$ in the region where $\hat{R}^{(N)}(\mu)$ gives the absolute minimum. The ratio of the thicknesses of two subsequent boundary layers becomes independent of n as well as independent of μ,

$$\frac{d_{n-1}^{(e)}}{d_n^{(e)}} = \left(\frac{b_{n+1}^2}{b_n b_{n-1}}\right)^{\frac{1}{3}} \mu^{(4-n)} = 4. \tag{4.10}$$

The thickness of the last innermost boundary layer is given by

$$d_N^{(e)} = (D/b_N\mu)^{\frac{1}{3}} = (\beta 4^{\frac{7}{3}}/\mu)^{\frac{1}{3}} \tag{4.11}$$

and the corresponding horizontal wavelength is

$$l_N^{(e)} = 2\pi/\alpha_N^{(e)} = 2\pi/(\sigma^{\frac{3}{2}}\beta^{-\frac{5}{2}}4^{\frac{4}{3}}\mu/4)^{\frac{1}{3}}. \tag{4.12}$$

Thus the scale of the bounding field decreases by steps with a factor of $\frac{1}{4}$ from a value of the order one in the interior to a value of the order $\mu^{-\frac{1}{3}}$ at the boundary.

The fact that the boundary layers fill a considerable portion of the interior according to (4.10) and that the thicknesses of two subsequent boundary layers differ only by a factor of four indicates that the boundary-layer approximation of the bounding field does not improve as μ tends to infinity. Because of the double limit of N as well as μ tending to infinity, an asymptotic case in which the boundary approximation of the minimizing multi α solution becomes exact does not exist. We consider, however, the deviations from the assumptions on which the boundary-layer analysis is based as not so dangerous that the derived results become qualitatively incorrect. A justification for this view is the fact that the upper estimate for $R^{(N)}(\mu)$ mentioned above does not diverge when N tends to infinity at a given value of μ. It differs only by a factor of about 2 from the asymptotic expression (4.2).

5. A necessary condition for the bounding field

In order to illustrate the physical meaning of the solutions described in the preceding sections we shall present a consideration by which it was originally deduced that the single α solution $v^{(1)}$, $\theta^{(1)}$ cannot describe the upper bound for large Rayleigh numbers. Let v_0, θ_0 be a solution of the Euler equations corresponding to a minimum of the functional (2.6). For simplicity we assume that $R_0(\mu)$ is proportional to μ^m for large μ with $m < 1$ and that accordingly $\overline{w_0\theta_0}$ differs from its average value unity essentially only over a distance $d\mu^{m-1}$ from the boundaries. For convenience we assume that the second normalization condition in (2.11) is replaced by

$$\langle|\nabla\theta_0|^2\rangle = -\langle v_0\Delta\Delta\Delta_2 v_0\rangle \equiv D_0\mu^{\frac{1}{2}m},$$

D_0 and d are constants independent of μ. We now consider a perturbation v_1, θ_1 of the solution v_0, θ_0 with the property that all averages of mixed products like $\overline{w_0\theta_1}$ vanish. The equations for v_1, θ_1 become linear when the amplitude of the perturbation is sufficiently small:

$$\left.\begin{aligned} D_0\Delta\Delta\Delta_2 v_1 + \mu^{1-\frac{1}{2}m}(\overline{w_0\theta_0}-1)\Delta_2\theta_1 - R_0\mu^{-\frac{1}{2}m}\Delta_2\theta_1 &= 0,\\ D_0\Delta\theta_1 + \mu^{1-\frac{1}{2}m}(\overline{w_0\theta_0}-1)\Delta_2 v_1 - R_0\mu^{-\frac{1}{2}m}\Delta_2 v_1 &= 0.\end{aligned}\right\} \tag{5.1}$$

Close to the boundary the third term in these equations can be neglected in comparison with the second term, and the equations become essentially identical with those governing the marginal stability of a static layer heated from below. A solution of (5.1) with a characteristic scale given by the thickness $d\mu^{m-1}$ of the layer in which $\overline{w_0\theta_0}$ is less than unity can exist when

$$\mu^{2-m}/D_0^2 \gtrsim R_c\mu^{4-4m}/d^4, \tag{5.2}$$

where the critical Rayleigh number R_c is a constant. Relation (5.2) is satisfied for sufficiently large values of μ when $m > \frac{2}{3}$. In this case a new solution $v_0+v_1, \theta_0+\theta_1$ of the Euler equations is possible, which for small but finite amplitude of v_1, θ_1 leads to a value of $R(\mu)$ below $R_0(\mu)$, as can be shown by a more detailed analysis. This 'instability' of v_0, θ_0 as extremalizing solution is very similar to the process by which $v^{(N)}, \theta^{(N)}$ is superseded by $v^{(N+1)}, \theta^{(N+1)}$ as extremalizing solution. Using a physical terminology the latter process could be called a 'finite amplitude instability'.

The above arguments as well as the actual solution $\hat{R}^{(\infty)}(\mu)$ derived in § 4 prove that $R(\mu)$ should grow like $\mu^{\frac{2}{3}}$ or less asymptotically. The latter possibility, however, is excluded by the exact solution (4.3) of the variational problem without the constraint of the equation of continuity. The fact that $\hat{R}^{(\infty)}(\mu)$ differs only by a relatively small factor from the expression (4.3) is another argument for our conjecture that the absolute minimum $R(\mu)$ of the functional (2.6) is described by the class of functions $\{R^{(N)}(\mu)\}$ corresponding to the multi α solutions.

6. Comparison with experimental observations

The theory outlined in the preceding sections gives bounds for the heat transport by turbulent convection. The fields $\mathbf{v}$ and θ have been introduced solely as mathematical tools for the solution of the variational problem. In principle the extremalizing solution can be disregarded after the bound on the heat transport has been obtained. The structure of the bounding field, however, seems to have so much physical content that it becomes an attractive idea to compare it with the structure of the observed turbulent convection.

There are also some rational reasons for this comparison. From the analysis the importance of the horizontal scale of the convection has become evident. A certain value of the heat transport imposes restrictions on the spectrum of convective motions by which the heat can be transported at a given Rayleigh number. In the limit when the convective heat transport vanishes at the critical value R_c of the Rayleigh number, the horizontal wave-number and even the z dependence of the realized convection are the same as for the bounding field. At higher Rayleigh number the observed heat transport is relatively close to the upper bound although the dependence of the Nusselt number appears to tend towards $Ra^{\frac{1}{3}}$ rather than $Ra^{\frac{1}{2}}$. These facts and others which will become evident in the further discussion indicate a close correlation between the experimentally observed convection and the solution of the variational problem.

The correlation appears striking enough to suggest a common principle behind the extremalizing multi α solution and the observed convection. Such a principle is the hypothesis that the realized solution of the Boussinesq equations is identical with the solution of maximum heat transport at a given Rayleigh number. This hypothesis was introduced by Malkus (1954*b*) in the theoretical analysis and was used later by Malkus & Veronis (1958) as a stability criterion. Although the hypothesis cannot be regarded as an exact general principle, it seems to be a rather good characterization of the realized convection. For Rayleigh numbers close to the critical value R_c at which the static layer becomes convectively unstable, the hypothesis has been proved (Busse 1967*b*) to hold exactly, provided that the Boussinesq equations with constant expansion coefficient are assumed.

The physical realization of a solution of the basic equations depends on its stability. In order to include non-stationary solutions of statistically stationary nature, we use the term 'stability' for stability of the structure of the solution as reflected in its r.m.s. values. The fact that the property of stability is closely related to the heat transport can be understood when the stability of the region close to the boundary is considered. In this boundary layer, in which the heat is transported mainly by conduction, the velocity tends to zero. Hence it is not unreasonable to apply as a crude qualitative criterion for the stability of this layer the Rayleigh stability criterion for a static layer heated from below. Accordingly, the layer with the thickness δ will become unstable with respect to disturbances of the characteristic scale δ when the dimensionless gradient of the mean temperature exceeds the value $R_c \delta^{-4}$. Since the change of mean temperature occurs mainly in the layer of thickness δ, the criterion for instability can be expressed by the dimensionless heat transport H, $\delta = Ra/2H$. The layer will become unstable unless the heat transport exceeds the lower bound:

$$H > \frac{Ra}{2}\left(\frac{Ra}{2R_c}\right)^{\frac{1}{3}}. \tag{6.1}$$

The conjecture that the physically realized convection is determined by the stability of the boundary layer is supported by measurements of the heat transport by Malkus (1954*a*) which show a sequence of kinks in the otherwise smooth dependence on the Rayleigh number. More recently the experiments have been repeated by Willis & Deardorff (1967). The phenomenon has to be interpreted as a sequence of instabilities, by which the convection changes its boundary-layer structure in discontinuous steps. The instability of the convection in the form of rolls (Busse 1967*a*) probably corresponds to one of the kinks. We note the qualitative similarity between the heat transport curve and the upper bound, both characterized by kinks. The process by which the absolute extremum of the variational functional is reached by one solution after the other of the multi α solutions is closely related to a sequence of finite amplitude boundary-layer instabilities as has been indicated earlier in the preceding section.

Owing to the boundary-layer approximation the description of the bounding field $v^{(e)}, \theta^{(e)}$ is incomplete and not quite suitable for a quantitative comparison with the measured r.m.s. values of the velocity components and the fluctuating temperature field. With respect to qualitative features, the similarity between the bounding field and the experimental data by Deardorff & Willis (1967; hereafter referred to as (DW)) is quite obvious. This fact is also apparent in the comparison in (H) between Townsend's (1959) measurements and the boundary-layer structure of the solution $v^{(1)}, \theta^{(1)}$. For the multi α solutions the decreasing dependence of $\theta^{(N)}$ with distance from the boundary is less pronounced and closer to the observed dependence of the temperature fluctuations.

A characteristic property of the bounding field is the set of discrete wavenumbers describing the horizontal dependence. At first sight this property appears as a typical unphysical feature, since turbulence is usually assumed to be associated with smooth spectra. The measurements of the spectrum of horizontal wavelengths λ in (DW), however, exhibit at least two characteristic lines besides a smooth background spectrum. In figure 3 we compare the experimental results obtained in the centre of the convection layer with the corresponding wave-

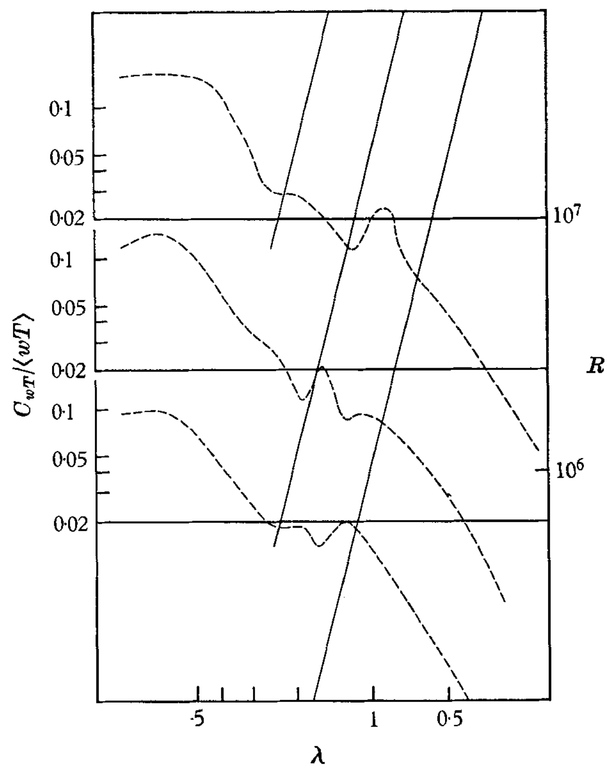

FIGURE 3. Three graphs of the normalized cospectra of w and θ, observed by Deardorff & Willis (1967) at Rayleigh numbers $6{\cdot}3\times10^5$, $2{\cdot}5\times10^6$, $1{\cdot}0\times10^7$ respectively, are plotted on top of a figure showing $l_1^{(N)}$ as a function of the Rayleigh number for $N = 2, 3, 4$. The three graphs have been arranged in such a way that the Rayleigh numbers of both plots coincide approximately at the level where the secondary maxima appear in the cospectra.

lengths $l_1^{(N)} = 2\pi/\alpha_1^{(N)}$ of the extremalizing multi α solutions. The lines $l_1^{(N)}(Ra)$ are plotted for $N = 2, 3, 4$ starting at about the Rayleigh number, where $\hat{\mu}^{(N)}(Ra)$ begins to represent the upper bound for the heat transport. On top of this graph data from (DW) taken at three different Rayleigh numbers have been plotted in such a way that the Rayleigh numbers of both graphs coincide at about the level where the secondary maxima appear. The data suggest that certain wavelengths are distinguished. They seem to appear on the left side of the spectrum and to become shorter with increasing Rayleigh number until they disappear on the right side of the spectrum. The figure indicates the close relationship to corresponding behaviour of the interior wavelength of the extremalizing solution. The appearance of more than one of those characteristic wavelengths does not contradict the conjecture that turbulent convection tends to adjust to the structure of the extremalizing solution. Since the values $\hat{\mu}^{(N-1)}(Ra)$ and $\hat{\mu}^{(N+1)}(Ra)$ are very close to the maximizing value $\hat{\mu}^{(N)}(Ra)$, the turbulent convection tends to adjust alternately to the extremalizing and its neighbouring solutions.

This interpretation of the occurrence of typical horizontal scales also explains the following paradox which has been discussed in papers on the convection problem. While the theoretical solutions always exhibit a horizontal scale which decreases with increasing Rayleigh number, a constant or slightly increasing horizontal scale is usually reported in experimental observations. The mechanism of instabilities by which new forms of convection with larger scales are introduced may be the answer to this paradox. The bounding field exhibits this behaviour. On an average, the interior wavelength stays approximately constant although the individual wavelengths $l_n^{(N)}$ always decrease.

It is of interest to note also the feature with respect to which the extremalizing solution differs from the experimentally observed fields. The time dependence, of course, of the turbulent convection does not have an analogue. Since only time-independent terms appear in the variational problem, the time dependence of the extremalizing solution is left undetermined. Another difference is the rather low correlation between the temperature field and the vertical velocity. According to measurements reported by (DW), the correlation between θ and w is only about 58 % while the maximum heat transport corresponds to a correlation of 100 %. For this reason quantities like the kinetic energy, or the average of the square of the temperature fluctuations which do not depend on the correlation, show a closer relation to the corresponding properties of the bounding field than does the heat transport. The qualitative difference between the measured dependence of the heat flux on the Rayleigh number and the $Ra^{\frac{1}{2}}$ law of the upper bound may be due to the fact that only rather low Reynolds numbers of the convection velocity have been realized in the experiment. For this reason the asymptotic region has probably not yet been reached. This view is in keeping with arguments used by Kraichnan (1962) who proposes a dependence of the form $Ra^{\frac{1}{2}}/\ln Ra$ for the Nusselt number.

The temperature field of a turbulent fluid without any heat sources has the property that maximum and minimum value of the temperature are attained at the boundary. This condition has not been imposed as additional constraint in the formulation (2.6) of the variational problem. It can be shown that the solutions $v^{(N)}, \theta^{(N)}$ in fact violate this condition for sufficiently high values of μ in the case $N > 1$. In the range of μ, however, where $\hat{R}(\mu) = \hat{R}^{(N)}(\mu)$ holds, the temperature corresponding to the function $\theta^{(N)}$ does not exceed the values at the boundaries. The analogue of the mean temperature gradient in the interior for the multi α solutions is given according to (3.33) by

$$\frac{dT^{(N)}}{dz} = \frac{-1}{3 \cdot 4^N - 1} \hat{R}^{(N)}(\mu). \qquad (6.2)$$

Using (4.2), (4.7) we find the following asymptotic dependence for the analogue of the interior mean temperature gradient in the case of the bounding field:

$$\frac{dT^{(e)}}{dz} = -\hat{R}^{\frac{1}{2}}(\beta 4^{\frac{4}{3}})^2/\sqrt{3}. \qquad (6.3)$$

At Rayleigh numbers sufficiently high to produce a fully developed boundary-layer structure, it is difficult to observe experimentally a gradient of the mean temperature as small as that given by (6.3). For this reason data for a comparison with this property of the bounding field do not exist.

7. Conclusion

The multi α solutions which have been derived to obtain an upper bound for the convective heat transport apparently have physical significance beyond this purpose. The structure and the dependence of the bounding field on the Rayleigh number offer a guide line for the understanding of the differentiated structures and processes involved in the complex phenomenon which is called thermal turbulence. It will be shown in a forthcoming paper by applying similar methods to shear flow problems that the correspondence between the extremalizing solution of a variational problem and experimental observations holds in many cases of turbulence. The mathematical picture of turbulence as a manifold of solutions of the basic equations which tends to approach the idealized structure with the property of optimal transport is in many respects complementary to the assumption of randomness used in the statistical description of turbulence. Both views are reflected in the nature of the observed turbulence, and theories evolving from both sides may be able to close the gap in the theoretical description of turbulence.

This work was started when the author participated in the Woods Hole Summer Programme in Geophysical Fluid Dynamics in 1967 under the directorship of George Veronis. The author is grateful for the stimulation he received from discussions with Louis N. Howard and Willem V. R. Malkus.

Appendix

The linear equations (3.23) can be solved readily when the corresponding non-linear boundary conditions at the point where $\hat{\Omega}\hat{\Theta}$ approaches unity are known. These boundary conditions are determined by the solution of (3.24) in the adjacent region. Since the general solution of (3.24) is not known, we have to use an approximation. Because $\hat{\Omega}$ has to become a linear function asymptotically

$$\hat{\Omega} \approx A(\xi-\xi_0) \quad \text{for} \quad \xi \to \infty, \tag{A 1}$$

we use an asymptotic expansion in powers of $(\xi-\xi_0)^{-1}$. From (3.19) we obtain the coefficients in this expansion:

$$\hat{\Omega} = A(\xi-\xi_0)\left[1-\frac{1}{12A^4(\xi-\xi_0)^2}-\frac{1}{270A^8(\xi-\xi_0)^4}-\frac{293}{907200A^{12}(\xi-\xi_0)^6}-\dots\right]. \tag{A 2}$$

At the point ξ^* where the solution (A 2) of (3.24) joins the solution of (3.23), $\hat{\Omega}$ and its derivatives up to the third derivative have to be continuous. This requirement together with the condition that $(\hat{\Omega}\hat{\Theta})'$ has to vanish at $\xi = \xi^*$ gives the following boundary conditions:

$$\left.\begin{aligned}\hat{\Omega}'' &= -\frac{1}{6\hat{\Omega}^3}\left(1+\frac{1}{60\hat{\Omega}^2\hat{\Omega}'^2}+\frac{1}{840\hat{\Omega}^4\hat{\Omega}'^4}+\ldots\right),\\ \hat{\Omega}''' &= \frac{1}{2\hat{\Omega}^4}\left(1+\frac{1}{36\hat{\Omega}^2\hat{\Omega}'^2}-\frac{11}{270\hat{\Omega}^4\hat{\Omega}'^4}+\ldots\right),\\ \hat{\Omega}'\hat{\Theta}+\hat{\Theta}'\hat{\Omega} &= 0.\end{aligned}\right\} \text{ at } \xi = \xi^*. \tag{A 3}$$

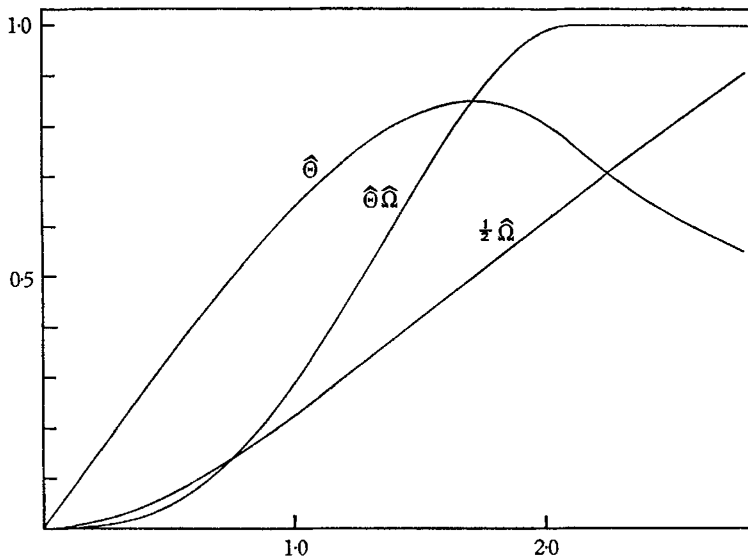

FIGURE 4. The boundary-layer solutions $\hat{\Omega}$, $\hat{\Theta}$ and their product.

In addition the solution has to be normalized to give $\hat{\Omega}(\xi^*).\hat{\Theta}(\xi^*) = 1$. We eliminate $\hat{\Theta}$ in (3.23) and obtain as general solution of the sixth-order equation

$$\hat{\Omega} = \sum_{\nu=1}^{3} (A_\nu \sin \omega_\nu \xi + B_\nu \cos \omega_\nu \xi), \tag{A 4}$$

where ω_ν are the three complex roots of $\omega^3 = 1$. The boundary conditions at $\xi = 0$ and at $\xi = \xi^*$ lead to 6 homogeneous equations which can be solved when ξ^* is chosen appropriately. The lowest value for which a solution can be obtained is $\xi^* = 2{\cdot}1025$. The corresponding values of A_ν and B_ν are

$$\left.\begin{aligned} A_1 &= 0{\cdot}58665, & B_1 &= -0{\cdot}41406,\\ A_2 &= 0{\cdot}24532+i0{\cdot}19706, & B_2 &= 0{\cdot}20703-i0{\cdot}35858,\\ A_3 &= \bar{A}_2, & B_3 &= \bar{B}_2,\end{aligned}\right\} \tag{A 5}$$

where in this case the bar denotes the complex conjugate.

For $\xi > \xi^*, \hat{\Omega}$ and $\hat{\Theta} = \hat{\Omega}^{-1}$ are determined by (A 2) with

$$A = 0{\cdot}6837, \quad \xi^* - \xi_0 = 2{\cdot}094. \tag{A 6}$$

The functions $\hat{\Omega}$, $\hat{\Theta}$, $\hat{\Omega}\hat{\Theta}$ have been plotted in figure 4.

According to (3.23), (3.24) the integrals of $\hat{\Theta}'^2$ and $\hat{\Omega}''^2$ should be equal. The higher derivatives, however, are rather sensitive to small changes in the approximation. For this reason the integrals

$$\int_0^\infty \hat{\Theta}'^2 \, d\xi = 0{\cdot}606, \quad \int_0^\infty \hat{\Omega}''^2 \, d\xi = 0{\cdot}593 \tag{A 7}$$

differ by more than one percent. For the numerical value of (3.25) we have used the mean value of (A 7) and the more exactly determined value of

$$\int_0^\infty (1 - \hat{\Omega}\hat{\Theta}) \, d\xi = 1{\cdot}248.$$

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